Stationary analysis of one channel queueing systems

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1 Introduction

In this text, I will analyze some properties of M/M/1 queueing systems.

Using Kendall's notation[2], an M/M/c queueing system consists of:

- a memoryless arrival process, modeled as Poisson process with rate λ, and
- c servers, each with a memoryless service time distribution, modeled as exponential distribution with rate μ.

We will only analyze systems with $c = 1$, i.e. one server.

Another parameter of the queueing systems we will consider is their capacity, i.e. the maximal number of customers allowed in the system. Let's denote this parameter as K.

Altogether, we have three parameters of interest:

- K: capacity
- λ: arrival rate
- μ: service rate

The state of the queueing system is the number of customers currently in the system (including those being served at the moment and those waiting in the queue). This means that a queueing system has exactly K states.

In every time $t \geq 0$, the queueing system is in one of the states. As time t increases, the system may change state according to arrival process and service process. The number of customers in the system increases by one when new customer arrives (unless the capacity is full) and decreases by one when a customer service finishes.

In the remainder of this text, we will perform stationary analysis of the system, i.e. analyze the system in limit time ($t \to \infty$).

2 Analysis

In this section, we will derive the stationary state distribution and expected state of a queuing system parametrized by K, λ and μ. We will consider both finite and infinite K.

2.1 Distribution of states

Let π_i denote the probability that the queueing system is in state i in limit time ($t \to \infty$).

The following equation must hold for each pair of states $i, i+1$ ($i < K$):

$$\pi_i \lambda = \pi_{i+1} \mu \tag{1}$$

The equation connects pairs of neighboring states.

Rewriting this equation we get a formula we can use to calculate π_{i+1} from π_i:

$$\pi_{i+1} = \pi_i \frac{\lambda}{\mu} \tag{2}$$

Let ρ denote $\frac{\lambda}{\mu}$. (We will continue to use this notation in the remainder of the text.)

Using induction, we get a formula that describes π_i as a function of π_0, ρ and i:

$$\pi_i = \pi_0 \rho^i \tag{3}$$

This gives us a system of K equations that describe π_i as a function of π_0, i and ρ for every $i \in \{1, \ldots, K\}$.

To finish the analysis, we need to restrict the sum of probabilities:

$$\sum_{i=0}^{K} \pi_i = 1 \tag{4}$$

Using formula 3, we get:

$$\sum_{i=0}^{K} \pi_0 \rho^i = 1 \tag{5}$$

$$\pi_0 \sum_{i=0}^{K} \rho^i = 1 \tag{6}$$

$$\pi_0 = \left(\sum_{i=0}^{K} \rho^i \right)^{-1} \tag{7}$$

We use formula 3 again to obtain a general equation:

$$\pi_i = \left(\sum_{i=0}^{K} \rho^i\right)^{-1} \rho^i \tag{8}$$

This is a general formula that holds for all values of K and ρ. In the following sections, we will express the sum $\sum_{i=0}^{K} \rho^i$ to obtain nicer formulas separately for various cases of K and ρ.

2.1.1 Limited capacity

Let $K \in \mathbb{N}$.

Let's first consider the case $\rho = 1$ (i.e. $\lambda = \mu$).

$$\sum_{i=0}^{K} \rho^i = \sum_{i=0}^{K} 1 = K + 1 \tag{9}$$

$$\begin{aligned} \pi_i &= \left(\sum_{i=0}^{K} \rho^i\right)^{-1} \rho^i && (10) \\ &= (K+1)^{-1} 1 && (11) \\ &= \frac{1}{K+1} && (12) \end{aligned}$$

This means that if arrival and service rates are equal and the queue is limited, the stationary distribution of states is uniform.

Now let's consider $\rho \neq 1$. Since ρ^i as a function of $i \in \{0, \ldots, K\}$ is a geometric series, we can use a formula[1, section Formula] to express the sum:

$$\sum_{i=0}^{K} \rho^i = \frac{1 - \rho^{K+1}}{1 - \rho} \tag{13}$$

Replacing the sum in formula 8 we get:

$$\begin{aligned} \pi_i &= \left(\sum_{i=0}^{K} \rho^i\right)^{-1} \rho^i && (14) \\ &= \left(\frac{1 - \rho^{K+1}}{1 - \rho}\right)^{-1} \rho^i && (15) \\ &= \frac{1 - \rho}{1 - \rho^{K+1}} \rho^i && (16) \end{aligned}$$

Namely:

$$\pi_0 = \frac{1 - \rho}{1 - \rho^{K+1}} \tag{17}$$

2.1.2 Unlimited capacity

Let $K = \infty$.

If $\rho \geq 1$, the queue grows indefinitely and the system doesn't have a stationary distribution.

Let $\rho < 1$.

We can express the sum $\sum_{i=0}^{K} \rho^i = \sum_{i=0}^{\infty} \rho^i$ using polylogarithm of order 0:

$$\begin{aligned} \sum_{i=0}^{\infty} \rho^i &= 1 + \sum_{i=1}^{\infty} \rho^i && (18) \\ &= 1 + Li_0(\rho) && (19) \\ &= 1 + \frac{\rho}{1-\rho} && (20) \\ &= \frac{(1-\rho)+\rho}{1-\rho} && (21) \\ &= \frac{1}{1-\rho} && (22) \end{aligned}$$

In equation 20 we used expression of Li_0 shown in [4, section Particular values].

Replacing the sum in formula 8 we get:

$$\begin{aligned} \pi_i &= \left(\sum_{i=0}^{K} \rho^i \right)^{-1} \rho^i && (23) \\ &= \left(\frac{1}{1-\rho} \right)^{-1} \rho^i && (24) \\ &= (1-\rho)\rho^i && (25) \end{aligned}$$

This shows that the distribution of states is geometric with success probability $1 - \rho$.

Namely:

$$\pi_0 = 1 - \rho \tag{26}$$

2.2 Expected number of customers

Let $E[I]$ denote the expected number of customers (i.e. state) of the queueing system in limit time.

$$E[I] = \sum_{i=0}^{K} i\pi_i \tag{27}$$

$$= \sum_{i=0}^{K} i\pi_0\rho^i \tag{28}$$

$$= \pi_0 \sum_{i=0}^{K} i\rho^i \tag{29}$$

Again, we will discuss cases of finite and infinite K separately.

2.2.1 Limited capacity

Let $K \in \mathbb{N}$.

Let $\rho = 1$.

$$E[I] = \pi_0 \sum_{i=0}^{K} i\rho^i \tag{30}$$

$$= \frac{1}{K+1} \sum_{i=0}^{K} i \tag{31}$$

$$= \frac{1}{K+1} \frac{(K+1)K}{2} \tag{32}$$

$$= \frac{K}{2} \tag{33}$$

Let $\rho \neq 1$.

$$\sum_{i=0}^{K} i\rho^i = \sum_{i=1}^{K} i\rho^i \tag{34}$$

$$= \rho \frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)^2} \tag{35}$$

Expression 35 comes from [3, section Low-order polylogarithms]. Using equation 29 and the value of pi_0 from equation 17 we get:

$$E[I] = \pi_0 \sum_{i=0}^{K} i\rho^i \tag{36}$$

$$= \frac{1-\rho}{1-\rho^{K+1}} \rho \frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)^2} \tag{37}$$

$$= \rho \frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)(1-\rho^{K+1})} \tag{38}$$

2.2.2 Unlimited capacity

Let $K = \infty$.

If $\rho \geq 1$, the queue grows indefinitely and the system doesn't have a stationary distribution.

Let $\rho < 1$.

We can express the sum $\sum_{i=0}^{K} i\rho^i = \sum_{i=0}^{\infty} i\rho^i$ using polylogarithm of order -1:

$$\begin{aligned}\sum_{i=0}^{\infty} i\rho^i &= \sum_{i=1}^{\infty} i\rho^i && (39)\\ &= Li_{-1}(\rho) && (40)\\ &= \frac{\rho}{(1-\rho)^2} && (41)\end{aligned}$$

In equation 41 we used expression of Li_{-1} shown in [4, section Particular values].

Applying equations 26 and 41 in equation 29 we get:

$$\begin{aligned}E[I] &= \pi_0 \sum_{i=0}^{K} i\rho^i && (42)\\ &= (1-\rho)\frac{\rho}{(1-\rho)^2} && (43)\\ &= \frac{\rho}{1-\rho} && (44)\end{aligned}$$

3 Conclusions

We have derived the following expressions for stationary state probability distribution and expected state of single channel queueing systems:

K	ρ	π_i	π_0	$E[I]$
$K \in \mathbb{N}$	$\rho = 1$	$\frac{1}{K+1}$	$\frac{1}{K+1}$	$\frac{K}{2}$
$K \in \mathbb{N}$	$\rho \neq 1$	$\frac{1-\rho}{1-\rho^{K+1}}\rho^i$	$\frac{1-\rho}{1-\rho^{K+1}}$	$\rho\frac{1-(K+1)\rho^K+K\rho^{K+1}}{(1-\rho)(1-\rho^{K+1})}$
$K = \infty$	$\rho \geq 1$	0	0	∞
$K = \infty$	$\rho < 1$	$(1-\rho)\rho^i$	$(1-\rho)$	$\frac{\rho}{1-\rho}$

A Sage implementation

To test the derived expressions on some small parameter values, I implemented Sage[1] functions `pi` and `ei` that compute values of π_i and $E[I]$ respectively.

[1] `http://www.sagemath.org/`

```
import math
def pi(i, rho, K):
    assert i >= 0
    assert rho >= 0
    assert K >= 0
    if math.isinf(K):
        if rho >= 1:
            return 0
        else:
            return (1 - rho) * rho^i
    else:
        if rho == 1:
            return 1 / (K+1)
        else:
            return (1 - rho) * rho^i / (1 - rho^(K+1))

import math
def ei(rho, K):
    assert rho >= 0
    assert K >= 0
    if math.isinf(K):
        if rho >= 1:
            return float("inf")
        else:
            return rho / (1 - rho)
    else:
        if rho == 1:
            return K / 2
        else:
            return pi(0, rho, K) * \
                rho * (1 - (K+1)*rho^K + K*rho^(K+1)) / \
                (1 - rho)^2
```

B Example solutions

K	ρ	π_0	$E[I]$
0	$\frac{1}{2}$	1	0
1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$
2	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{4}{7}$
∞	$\frac{1}{2}$	$\frac{1}{2}$	1
0	1	1	0
1	1	$\frac{1}{2}$	$\frac{1}{2}$
2	1	$\frac{1}{3}$	1
∞	1	0	$+infinity$
0	2	1	0
1	2	$\frac{1}{3}$	$\frac{2}{3}$
2	2	$\frac{1}{7}$	$\frac{10}{7}$
∞	2	0	$+infinity$

References

[1] Wikipedia. Geometric series — Wikipedia, The Free Encyclopedia, 2014. [Online; accessed 8-June-2014].

[2] Wikipedia. Kendall's notation — Wikipedia, The Free Encyclopedia, 2014. [Online; accessed 4-June-2014].

[3] Wikipedia. List of mathematical series — Wikipedia, The Free Encyclopedia, 2014. [Online; accessed 8-June-2014].

[4] Wikipedia. Polylogarithm — Wikipedia, The Free Encyclopedia, 2014. [Online; accessed 4-June-2014].